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Local convergence of the method of pseudolinear equations for quasilinear elliptic boundary-value problems *

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Abstract: A variant of the method of pseudolinear equations, an iterative method of solving quasilinear partial differential equations, is described for quasilinear elliptic boundary-value problems of the type $-[p_1(u_x)]_x - [p_2(u_y)]_y = f$ on a bounded simply connected two-dimensional domain D. A theorem on local convergence in $C^{2,\lambda}(\bar{D})$ of this variant, which has constant coefficients, is proved. Three other methods of solving quasilinear elliptic boundary-value problems, namely, Newton's method, the Kačanov method and a variant of the method of successive approximations that has constant coefficients, are briefly discussed. Results of a series of numerical experiments in a finite-difference setting of solving quasilinear Dirichlet problems of the above-mentioned type by the method of pseudolinear equations and these three methods are given. These results show that Newton's method converges for stronger nonlinearities than do the other methods, which, in order thereafter, are the Kačanov method, the method of pseudolinear equations and, last, the method of successive approximations, which converges only for relatively weak nonlinearities. From fastest to slowest, the methods are: the method of successive approximations, the method of pseudolinear equations, Newton's method, the Kačanov method.

Keywords: Kačanov method, Newton's method, pseudolinear equations, quasilinear elleptic partial differential equations, successive approximations.

AMS(MOS) Subject Classification: Primary 65N99, secondary 35J60, 49D99.

1. Introduction

In [4,5], an iterative method, which we here call the method of pseudolinear equations, for solving quasilinear elliptic boundary-value problems was proposed. The main objective of the present paper is to investigate the local convergence of a variant of this method that has constant coefficients and to compare this local convergence numerically with the local convergence of three other methods for solving quasilinear elliptic boundary-value problems, namely, Newton's

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method, the Kačanov method and the method of successive approximations.

We mention in passing that the method of pseudolinear equations has been adapted to solving quasilinear hyperbolic initial-boundary-value problems [6].

All quantities in this paper are real and scalar-valued.

We begin with a brief description of the concept of conjugate quasilinear Dirichlet and Neumann problems, on which the method of pseudolinear equations is based.

2. Conjugate quasilinear problems

Throughout this paper, we consider as our basic problem the Dirichlet problem of finding U in some function space such that

$$-[p_1(U_x)]_x - [p_2(U_y)]_y = f(x, y) \tag{2.1a}$$

in a bounded simply connected domain D in $\mathbb{R}^2$ with sufficiently smooth boundary ∂D and

$$U|_{\partial D} = g. \tag{2.1b}$$

We will assume that, for some positive constant A_u,

$$p_i \in C^1[-A_u, A_u], \quad i = 1, 2, \tag{2.2a}$$

$$p_i'(X) > 0, \quad X \in [-A_u, A_u], \; i = 1, 2, \tag{2.2b}$$

which implies that (2.1a) is elliptic as long as the arguments of the p_i remain within $[-A_u, A_u]$.

The Neumann problem conjugate to problem (2.1) is defined as follows. Let A_v be a positive constant such that

$$p_i([-A_u, A_u]) \subset [-A_v, A_v], \quad i = 1, 2. \tag{2.2c}$$

Assume that there exist 'quasi-inverse' functions

$$\hat{p}_1(Z) \equiv p_2^{-1}(Z), \qquad \hat{p}_2(Z) \equiv -p_1^{-1}(-Z), \quad Z \in [-A_v, A_v], \tag{2.2d}$$

such that

$$\hat{p}_i \in C^1[-A_v, A_v], \quad i = 1, 2, \tag{2.2e}$$

and

$$\hat{p}_i'(Z) > 0, \quad Z \in [-A_v, A_v], \; i = 1, 2. \tag{2.2f}$$

Let α and β be fixed functions on D such that

$$f = \alpha_x + \beta_y. \tag{2.3}$$

The Neumann problem conjugate to the Dirichlet problem (2.1) is the problem of finding V in some function space such that

$$-[\hat{p}_1(V_x - \beta)]_x - [\hat{p}_2(V_y + \alpha)]_y = 0 \tag{2.4a}$$

in D and

$$\nu_1 \hat{p}_1(V_x - \beta) + \nu_2 \hat{p}_2(V_y + \alpha) = \mathrm{d}g/\mathrm{d}s \tag{2.4b}$$

on ∂D, where (ν_1, ν_2) is the outward normal and $\mathrm{d}g/\mathrm{d}s$ is the arclength derivative of g in the counterclockwise direction. We normalize the solution V by requiring that, for some fixed point $(x_0, y_0) \in \overline{D}$,

$$V(x_0, y_0) = 0. \tag{2.4c}$$

Remark 2.1. The conjugate Neumann problem defined here differs from the conjugate Neumann problems defined in [4,5] only in notation. This difference in notation is due to the fact that the $\hat{p}_i$ of [4,5] are slightly different from the $\hat{p}_i$ of identities (2.2d).

If a solution U of problem (2.1) exists for which the values of $U_x(x, y)$ and $U_y(x, y)$ are in $[-A_u, A_u]$ $\forall (x, y) \in \overline{D}$, then a solution V of problem (2.4) exists for which $V_x(x, y) - \beta(x, y)$ and $V_y(x, y) + \alpha(x, y)$ are in $[-A_v, A_v]$ $\forall (x, y) \in \overline{D}$ and the relations

$$V_x = p_2(U_y) + \beta, \qquad V_y = -p_1(U_x) - \alpha, \tag{2.5a}$$

and, equivalently,

$$U_x = -\hat{p}_2(V_y + \alpha), \qquad U_y = \hat{p}_1(V_x - \beta) \tag{2.5b}$$

hold (cf. [4,5]). The proof of this assertion consists in defining V from U by (2.5a) and (2.4c) and using (2.5b) to show that the V thus defined satisfies (2.4a) and (2.4b).

3. The method of pseudolinear equations

The method of pseudolinear equations is based on relations (2.5). The variant of this method with constant coefficients can be introduced as follows. Let positive constants q and $\hat{q}$ be given. From (2.5a) and (2.4c), it is clear that U and V satisfy the equality

$$\begin{aligned} -\hat{q}V_{xx} - V_{yy} &= -\hat{q}\left[p_2(U_y) + \beta\right]_x + \left[p_1(U_x) + \alpha\right]_y \\ &= -\hat{q}\left[p_2'(U_y)U_{xy} + \beta_x\right] + p_1'(U_x)U_{xy} + \alpha_y \end{aligned} \tag{3.1a}$$

in D, the boundary condition

$$\nu_1\hat{q}V_x + \nu_2V_y = \nu_1\hat{q}\left[p_2(U_y) + \beta\right] - \nu_2\left[p_1(U_x) + \alpha\right] \tag{3.1b}$$

on ∂D and the normalization condition

$$V(x_0, y_0) = 0. \tag{3.1c}$$

Relations (2.5b) and (2.1b) imply that U and V also satisfy

$$\begin{aligned} -qU_{xx} - U_{yy} &= q\left[\hat{p}_2(V_y + \alpha)\right]_x - \left[\hat{p}_1(V_x - \beta)\right]_y \\ &= q\hat{p}_2'(V_y + \alpha)(V_{xy} + \alpha_x) - \hat{p}_1'(V_x - \beta)(V_{xy} - \beta_y) \end{aligned} \tag{3.2a}$$

in D and the boundary condition

$$U|_{\partial D} = g. \tag{3.2b}$$

Let there now be given an approximate solution $u^{(k)}$ of problem (2.1). We can calculate an approximate solution $v^{(k)}$ of the conjugate quasilinear Neumann problem (2.4) from $u^{(k)}$ by solving the following linear Neumann problem consisting of analogues of equalities (3.1):

$$-\hat{q}v_{xx}^{(k)} - v_{yy}^{(k)} = -\hat{q}\left[p_2'\left(u_y^{(k)}\right)u_{xy}^{(k)} + \beta_x\right] + p_1'\left(u_x^{(k)}\right)u_{xy}^{(k)} + \alpha_y \tag{3.3a}$$

in D,

$$\nu_1\hat{q}v_x^{(k)} + \nu_2 v_y^{(k)} = \nu_1\hat{q}\left[p_2\left(u_y^{(k)}\right) + \beta\right] - \nu_2\left[p_1\left(u_x^{(k)}\right) + \alpha\right] \tag{3.3b}$$

on ∂D and

$$v^{(k)}(x_0, y_0) = 0, \tag{3.3c}$$

where (x_0, y_0) is the same fixed point in $\overline{D}$ as in (2.4c) and (3.1c). From $v^{(k)}$ we can calculate a new approximate solution $u^{(k+1)}$ of the quasilinear Dirichlet problem (2.1) by solving the following linear Dirichlet problem consisting of analogues of equalities (3.2):

$$-qu_{xx}^{(k+1)} - u_{yy}^{(k+1)} = q\hat{p}_2'\left(v_y^{(k)} + \alpha\right)\left(v_{xy}^{(k)} + \alpha_x\right) - \hat{p}_1'\left(v_x^{(k)} - \beta\right)\left(v_{xy}^{(k)} - \beta_y\right) \tag{3.4a}$$

in D and

$$u^{(k+1)}|_{\partial D} = g. \tag{3.4b}$$

Concerning the interpretation of problems (3.3) and (3.4) as problems of minimization of error in analogues of relations (2.5), see [4,5].

The method of pseudolinear equations consists in calculating the sequence

$$u^{(0)} \to v^{(0)} \to u^{(1)} \to v^{(1)} \to u^{(2)} \to \cdots \tag{3.5}$$

starting from a given $u^{(0)}$ satisfying boundary condition (3.4b) ($k+1=0$). Sufficient conditions that the iterates $u^{(k)}$ and $v^{(k)}$ of sequence (3.5) converge globally in the energy spaces to the solutions U and V of the quasilinear problems (2.1) and (2.4), respectively, are given in [4,5]. In the next section, we investigate the local convergence of the $u^{(k)}$ and the $v^{(k)}$ in $C^{2,\lambda}(\overline{D})$.

4. Local convergence of the method of pseudolinear equations

The theorem presented below for the variant of the method of pseudolinear equations with constant coefficients is indicative of the type of local convergence results that can be obtained for the method of pseudolinear equations in general. In the theorem, we will use the following elementary lemma.

Lemma 4.1. *For any* $X \in [-A_u, A_u]$ *and any* $Z \in [-A_v, A_v]$,

$$(Z + p_1(X))(\hat{p}_2(Z) + X) \geqslant 0, \qquad (Z - p_2(X))(\hat{p}_1(Z) - X) \geqslant 0.$$

Proof. By relations (2.2d), an intermediate-value theorem and (2.2f),

$$\begin{aligned}(Z + p_1(X))(\hat{p}_2(Z) + X) &= (Z + p_1(X))[\hat{p}_2(Z) - \hat{p}_2(-p_1(X))] \\ &= \hat{p}_2'(W)(Z + p_1(X))^2 \geqslant 0,\end{aligned}$$

where $W = -p_1(X) + t(Z + p_1(X))$ for some t, $0 < t < 1$. The second inequality is proved analogously. □

In what follows, the norm symbol without subscript designates the $C^{2,\lambda}(\bar{D})$ norm that consists of the sum of the maximum moduli of the function and its partial derivatives up to and including second order and of the Hölder constants of the second-order derivatives, that is,

$$\|\cdot\| \equiv \|\cdot\|_{C^{2,\lambda}(\bar{D})}.$$

Theorem 4.2. *Let λ and a_u be constants, $0<\lambda<1$, $0<a_u<A_u$. Let*

$$\|u^{(0)}\| \leqslant a_u. \tag{4.1}$$

Let $\alpha \in C^{1,\lambda}(\bar{D})$, $\beta \in C^{1,\lambda}(\bar{D})$, $\partial D \in C^{2,\lambda}$ and $g \in C^{2,\lambda}(\partial D)$. Let I be the identity function, that is, $I(X)=X$. Let $p_i \in C^{2,\lambda}[-A_u, A_u]$ and $\hat{p}_i \in C^{2,\lambda}[-A_v, A_v]$ be such that, for some constant c_p,

$$\begin{aligned} &\|p_1 - qI\|_{C^{2,\lambda}[-A_u,A_u]} \leqslant c_p, && \|p_2 - I\|_{C^{2,\lambda}[-A_u,A_u]} \leqslant c_p, \\ &\|\hat{p}_1 - I\|_{C^{2,\lambda}[-A_v,A_v]} \leqslant c_p, && \|\hat{p}_2 - I/\hat{q}\|_{C^{2,\lambda}[-A_v,A_v]} \leqslant c_p. \end{aligned} \tag{4.2}$$

Then, for fixed ∂D, α, β and q, if a_u, $|q-\hat{q}|$ and c_p are sufficiently small and A_u and A_v are sufficiently large, the iterates $u^{(k)}$ converge linearly in

$$C_u = \left\{ u \in C^{2,\lambda}(\bar{D}) : \|u_x\|_{C^0(\bar{D})} \leqslant A_u, \|u_y\|_{C^0(\bar{D})} \leqslant A_u \right\}$$

to a solution U of problem (2.1), *the $v^{(k)}$ converge linearly in*

$$C_v = \left\{ v \in C^{2,\lambda}(\bar{D}) : \|v_x - \beta\|_{C^0(\bar{D})} \leqslant A_v, \|v_y + \alpha\|_{C^0(\bar{D})} \leqslant A_v \right\}$$

to a solution V of problem (2.4) *and the pair $\{U, V\}$ is the unique pair of solutions of problems* (2.1) *and* (2.4) *in $C_u \times C_v$.*

Proof. The idea of the proof is to show that the arguments of the p_i and the $\hat{p}_i$ in (3.3) and (3.4) remain within the intervals $[-A_u, A_u]$ and $[-A_v, A_v]$ for all k and that, when this is the case, the mapping of $u^{(k)}$ into $u^{(k+1)}$ and the mapping of $v^{(k)}$ into $v^{(k+1)}$ are contraction mappings in C_u and C_v, respectively.

In what follows, we will use inequalities (1.11) and (3.7) of [3, p. 110 and p. 137] to estimate the $C^{2,\lambda}(\bar{D})$ norms of the solutions of the linear Dirichlet problem (3.4) and the linear Neumann problem (3.3). The maximum of the solution over $\bar{D}$, which appears as a term on the right-hand side of these inequalities, has been eliminated because the solutions of problems (3.3) and (3.4) are unique (cf. remark after inequality (1.11) of [3, p. 110]). Inequality 3.7 of [3], equalities (3.3) above and the conditions of the theorem imply that there exist constants c_1 and c_2 such that

$$\|v^{(0)}\| \leqslant c_1(a_u + c_2). \tag{4.3}$$

The constants c_i of inequalities (4.3) and all constants c_i below depend on ∂D, α, β, a_u, q, $\hat{q}$ and c_p. Inequality (1.11) of [3] with equalities (3.4) and inequality (4.3) above imply that

$$\|u^{(1)} - u^{(0)}\| \leqslant c_3\left[\|v^{(0)}\| + c_4 + \|u^{(0)}\|\right] \leqslant c_3\left[c_1(a_u + c_2) + c_4 + a_u\right] \tag{4.4}$$

for some constants c_3 and c_4. Subtracting (3.3) from the equalities (3.3) in which k is replaced by $k-1$, subtracting (3.4) from the equalities (3.4) in which k is replaced by $k-1$ and using the above-mentioned inequalities of [3], we obtain that there exist constants c_5 and c_6 such that

$$\|v^{(k)} - v^{(k-1)}\| \leqslant c_5 \|u^{(k)} - u^{(k-1)}\|, \tag{4.5a}$$

$$\|u^{(k+1)} - u^{(k)}\| \leqslant c_6 \|v^{(k)} - v^{(k-1)}\|. \tag{4.5b}$$

Inequalities (4.3)–(4.5) hold as long as the arguments of the p_i and the $\hat{p}_i$ remain within $[-A_u, A_u]$ and $[-A_v, A_v]$, respectively. In order to ensure that this is the case, we assume that a_u, $|q-\hat{q}|$ and c_p are sufficiently small and that A_u and A_v are sufficiently large so that

$$c_5 c_6 < 1, \tag{4.6}$$

$$a_u + c_3\left[c_1(a_u + c_2) + c_4 + a_u\right]/(1 - c_5 c_6) \leqslant A_u, \tag{4.7a}$$

$$c_1(a_u + c_2) + \frac{c_3 c_5\left[c_1(a_u + c_2) + c_4 + a_u\right]}{(1 - c_5 c_6)} + \max\left\{ \|\alpha\|_{C^0(\bar{D})}, \|\beta\|_{C^0(\bar{D})} \right\} \leqslant A_v. \tag{4.7b}$$

Having made the terms 'sufficiently small' and 'sufficiently large' precise in (4.6) and (4.7), we proceed to carrying out the proof of the theorem. Inequalities (4.3)–(4.7) imply that

$$\|u_x^{(k)}\|_{C^0(\bar{D})}, \|u_y^{(k)}\|_{C^0(\bar{D})} \leqslant \|u^{(k)}\| \leqslant \|u^{(0)}\| + \sum_{i=1}^{k} \|u^{(i)} - u^{(i-1)}\| \leqslant A_u, \tag{4.8a}$$

$$\begin{aligned} \|v_x^{(k)} - \beta\|_{C^0(\bar{D})}, \|v_y^{(k)} + \alpha\|_{C^0(\bar{D})} &\leqslant \|v^{(k)}\| + \max\left\{ \|\alpha\|_{C^0(\bar{D})}, \|\beta\|_{C^0(\bar{D})} \right\} \\ &\leqslant \|v^{(0)}\| + \sum_{i=1}^{k} \|v^{(i)} - v^{(i-1)}\| + \max\left\{ \|\alpha\|_{C^0(\bar{D})}, \|\beta\|_{C^0(\bar{D})} \right\} \leqslant A_v, \end{aligned} \tag{4.8b}$$

that is, the arguments of the p_i and the $\hat{p}_i$ remain within the intervals $[-A_u, A_u]$ and $[-A_v, A_v]$, respectively, for all k. Therefore, inequalities (4.3) and (4.4) hold and inequalities (4.5) hold uniformly for all k. But then (4.5) and (4.6) imply that the mapping $u^{(k)} \to u^{(k+1)}$ and the mapping $v^{(k)} \to v^{(k+1)}$ are contraction mappings in C_u and C_v, respectively. Hence, the $u^{(k)}$ converge to a function $U \in C_u$ and the $v^{(k)}$ converge to a function $V \in C_v$ and the pair $\{U, V\}$ is the unique pair of solutions of problems (3.1) and (3.2) in $C_u \times C_v$.

That the solutions U and V of problems (3.1) and (3.2) thus obtained are also solutions of problems (2.1) and (2.4) can be shown as follows. Equalities (3.1) imply that U and V satisfy the variational equality

$$\iint_D \left[\hat{q}\left(V_x - \beta - p_2(U_y)\right)h_x + \left(V_y + \alpha + p_1(U_x)\right)h_y\right] \mathrm{d}D = 0 \tag{4.9}$$

for all h in $H^1(D)$. In this equality, we will choose h such that

$$h_x = \hat{p}_1(V_x - \beta) - U_y, \qquad h_y = q\left(\hat{p}_2(V_y + \alpha) + U_x\right). \tag{4.10}$$

The necessary and sufficient condition that such an h, unique up to an arbitrary additive constant, exist is $(h_x)_y = (h_y)_x$, which holds by (3.2a). The integrand of (4.9) with the h of (4.10) consists of the two summands

$$\hat{q}\left(V_x - \beta - p_2(U_y)\right)\left(\hat{p}_1(V_x - \beta) - U_y\right), \qquad q\left(V_y + \alpha + p_1(U_x)\right)\left(\hat{p}_2(V_y + \alpha) + U_x\right).$$

Since both summands are, by Lemma 4.1, nonnegative and their sum is, by (4.9), identically zero, each of them must be identically zero, that is, (2.5a) (and, therefore, also (2.5b)) must hold.

Equalities (2.5) imply that U and V are solutions of the quasilinear problems (2.1) and (2.4). That the solutions of problems (2.1) and (2.4) are unique in $C_u \times C_v$ is again a consequence of (2.5). Assume, for example, that another solution $u \in C_u$ of problem (2.1) exists. Define $v \in C_v$ with $v(x_0, y_0) = 0$ from u via (2.5a) in which u and v replace U and V. Equalities (2.5) with these functions u and v instead of U and V then imply that u and v are also solutions of problems (3.1) and (3.2), which have only one pair of solutions $\{U, V\}$ in $C_u \times C_v$. The theorem is proved. □

5. Three other methods

Since we are interested mainly in solving problem (2.1) and not directly in solving the conjugate quasilinear problem (2.4), we describe here Newton's method, the Kačanov method and the method of successive approximations as applied to solving problem (2.1) only.

Newton's method (cf. [2, Ch. 18]) consists in the following. From a given approximation $u^{(k)}$ of the solution U of problem (2.1), compute the solution w of the linear Dirichlet problem

$$\begin{aligned} &-p_1'\big(u_x^{(k)}\big)w_{xx} - p_2'\big(u_y^{(k)}\big)w_{yy} - p_1''\big(u_x^{(k)}\big)u_{xx}^{(k)}w_x - p_2''\big(u_y^{(k)}\big)u_{yy}^{(k)}w_y \\ &\quad = f + p_1'\big(u_x^{(k)}\big)u_{xx}^{(k)} + p_2'\big(u_y^{(k)}\big)u_{yy}^{(k)} \end{aligned} \tag{5.1a}$$

in D and

$$w|_{\partial D} = 0. \tag{5.1b}$$

The next approximation $u^{(k+1)}$ of U is calculated by

$$u^{(k+1)} = u^{(k)} + w. \tag{5.2}$$

The Kačanov method [1] can be described as follows. Assume that there exist positive functions r_i such that

$$p_i(X) = r_i(X)X, \quad i = 1, 2.$$

From a given approximation $u^{(k)}$ of U, we calculate the next approximation $u^{(k+1)}$ by solving the linear problem

$$\left.\begin{aligned} &-r_1\big(u_x^{(k)}\big)u_{xx}^{(k+1)} - r_2\big(u_y^{(k)}\big)u_{yy}^{(k+1)} \\ &-r_1'\big(u_x^{(k)}\big)u_{xx}^{(k)}u_x^{(k+1)} - r_2'\big(u_y^{(k)}\big)u_{yy}^{(k)}u_y^{(k+1)} \end{aligned}\right\} = f \tag{5.3a}$$

in D and

$$u^{(k+1)}|_{\partial D} = g. \tag{5.3b}$$

Finally, we describe a variant of the method of successive approximations [7] that has constant coefficients. Let q_1 and q_2 be positive constants. From a given approximation $u^{(k)}$ of U, calculate the solution w of the linear Dirichlet problem

$$-q_1 w_{xx} - q_2 w_{yy} = f + p_1'\big(u_x^{(k)}\big)u_{xx}^{(k)} + p_2'\big(u_y^{(k)}\big)u_{yy}^{(k)} \tag{5.4a}$$

in D and

$$w|_{\partial D} = 0. \tag{5.4b}$$

The next approximation $u^{(k+1)}$ of U is then calculated by (5.2).

6. The numerical experiments and results

Theorem 4.2 of this paper and local convergence theorems for the three methods described in the preceding section contain sharp qualitative results but do not permit precise quantitative comparison of the four methods. The numerical results presented in this section, which give some idea of the behavior of the methods, fill this gap.

In all of the numerical experiments, D was taken to be the square $(0,1)\times(0,1)$. On $\overline{D}$, a uniform mesh of $(n+1)^2$ points with mesh length $h=1/n$ in both directions was placed. Problems (3.3), (3.4), (5.1), (5.3) and (5.4) were reduced to matrix equations by replacing all derivatives by their usual centered second-order finite-difference approximations at the nodes of the mesh. Thus, for any function z, the following substitutions were made:

$$z_x(x_i, y_j) \to (z_{i+1,j} - z_{i-1,j})/2h,$$

$$z_{xx}(x_i, y_j) \to (z_{i+1,j} - 2z_{i,j} + z_{i-1,j})/h^2,$$

analogously for $z_y(x_i, y_j)$ and $z_{yy}(x_i, y_j)$, and

$$z_{xy}(x_i, y_j) \to (z_{i+1,j+1} - z_{i+1,j-1} - z_{i-1,j+1} + z_{i-1,j-1})/4h^2.$$

The unknowns of the linear systems of equations that resulted were numbered row by row from bottom to top and, within each row, from left to right. With the normalization $v_{n+1,n+1}^{(k)} = 0$ from (3.3c), in which (x_0, y_0) is chosen to be (1,1), the Neumann problem (3.3) yields a system of equations for $(n+1)^2 - 1$ unknowns with an unsymmetric positive-definite matrix. The Dirchlet problems (3.4) and (5.4) yield systems of equations for $(n-1)^2$ unknowns with symmetric positive-define matrices. The linear Dirichlet problems (5.1) and (5.3) yield systems of equations for $(n-1)^2$ unknowns with unsymmetric positive-definite matrices.

Four computer programs for constructing and solving these linear systems of equations were written by the author while at the Computing Center of the Siberian Branch of the Academy of Sciences of the USSR in Novosibirsk. The numerical results presented below were obtained on a BESM-6 at this center.

The linear systems of equations were solved by system subroutines CHODET and CHOSOL (for symmetric systems) and BADET1 and BASOL1 (for unsymmetric systems) for the band Cholesky method. These subroutines are FORTRAN versions of the ALGOL procedures chobanddet, chobandsol, bandet1 and bansol1 of Wilkinson and Reinsch [8]. In all experiments, the values

$$n = 14, \qquad h = 1/14,$$

were used. Computation was carried out until one of the following conditions was satisfied:

$$\text{convergence} \begin{cases} \max\limits_{i,j} |v_{i,j}^{(k)} - v_{i,j}^{(k-1)}| < 10^{-6}, & \text{(6.1a)} \\ \max\limits_{i,j} |u_{i,j}^{(k+1)} - u_{i,j}^{(k)}| < 10^{-6}, & \text{(6.1b)} \end{cases}$$

$$\text{divergence} \begin{cases} \max\limits_{i,j} |v_{i,j}^{(k)} - v_{i,j}^{(k-1)}| \geqslant 10^{5}, & \text{(6.1c)} \\ \max\limits_{i,j} |u_{i,j}^{(k+1)} - u_{i,j}^{(k)}| \geqslant 10^{5}, & \text{(6.1d)} \end{cases}$$

pendulation or very slow convergence or divergence $\quad k+1=k_{\text{limit}}.$ (6.1e)

Conditions (6.1a) and (6.1c) apply to the method of pseudolinear equations only. Condition (6.1e) was tested after computation of $u^{(k+1)}$. The maximum number of iterations k_{limit} of this condition was taken to be 29 for Newton's method and the Kačanov method, 199 for the method of successive approximations and 49 for the method of pseudolinear equations.

In all experiments, the functions

$$g=0 \tag{6.2a}$$

and

$$p_i(X)=\left(1+c_iX^2\right)X, \quad i=1,2, \tag{6.2b}$$

(c_i = constant) were used. The functions α and β for problems (3.3) and (3.4) were taken to be

$$\alpha(x,y)=\int_0^x f(\xi,y)\,\mathrm{d}\xi, \qquad \beta(x,y)\equiv 0, \quad (x,y)\in\overline{D}.$$

The values of the functions $\hat{p}_i$ for problem (3.4) were calculated by Newton's method for scalar-valued functions of one real variable. The initial iterate for calculating $\hat{p}_i(Z)$ was Z and the calculations were carried out until the difference between two successive iterates was $\leqslant 10^{-8}$. The Cardano formula, by which the $\hat{P}_i$ can be explicitly expressed, was not used because it is much slower than Newton's method.

Selected numerical results are presented in Tables 1–5. In each box in these tables, the following data are given:

(1) the last iterate $v^{(k)}$ or $u^{(k+1)}$ computed;

(2) The CPU time quantified in units of 0.02 sec, inclusive of time for printing certain parameters;

(3) an appropriate word or phrase describing the behavior of the iterates – 'converges' (inequality (6.1a) or (6.1b) is satisfied), 'diverges' (inequality (6.1c) or (6.1d)) or 'it. limit' ('iterations reached limit', equality (6.1e)).

In Tables 1 and 2, no results for the Kačanov method are given because the function f of these tables is identically zero. This implies that the Kačanov method converges to the exact solution U, which is also identically zero, on the first iteration. This situation is, however, not typical for the method.

The following tendencies can be noted in the data in Tables 1–4. Newton's method converges for stronger nonlinearities, that is, for larger c_i and c, than do the other three methods. Next best is the Kačanov method, then the method of pseudolinear equations and last the method of successive approximations, which converges only for relatively weak nonlinearities. As regards computing time, the situation is as follows. The method of successive approximations takes far less CPU time than does any of the other three methods. For those entries of Tables 1 and 2 for which all three methods converge and those entries of Tables 3 and 4 for which all four methods converge, the method of pseudolinear equations takes on the average 3.6 times as much CPU time as does the method of successive approximations and Newton's method takes 4.7 times as much. For those entries of Tables 3 and 4 for which all four methods converge, the Kačanov method takes on the average 5.2 times as much CPU time as the method of successive approximations.

That the method of successive approximations and the method of pseudolinear equations are

Table 1
Numerical results for $f = 0$, $u^{(0)}(x, y) = c(x - x^2)(y - y^2)$

c_1	c_2	c	Newton	Successive approximations ($q_1 = q_2 = 1$)	Pseudolinear equations ($q = \hat{q} = 1$)
0.01	0.01	10	$u^{(4)}$ 32.60 sec converges	$u^{(3)}$ 3.84 sec converges	$u^{(2)}$ 20.04 sec converges
		100	$u^{(7)}$ 56.62 sec converges	$u^{(3)}$ 4.10 sec diverges	$u^{(3)}$ 22.88 sec converges
0.1	0.1	10	$u^{(5)}$ 38.68 sec converges	$u^{(5)}$ 5.00 sec converges	$v^{(2)}$ 20.92 sec converges
		100	$u^{(9)}$ 70.38 sec converges	$u^{(2)}$ 3.20 sec diverges	$v^{(2)}$ 21.20 sec diverges
1	1	10	$u^{(7)}$ 52.80 sec converges	$u^{(3)}$ 3.76 sec diverges	$u^{(3)}$ 21.42 sec converges
		100	$u^{(12)}$ 98.56 sec converges	$u^{(2)}$ 3.28 sec diverges	$v^{(1)}$ 18.60 sec diverges
10	10	10	$u^{(9)}$ 67.92 sec converges	$u^{(2)}$ 3.16 sec diverges	$v^{(3)}$ 24.06 sec diverges
		100	$u^{(15)}$ 124.80 sec converges	$u^{(2)}$ 3.18 sec diverges	$v^{(1)}$ 18.52 sec diverges

faster than Newton's method and the Kačanov method is due in part to the fact that for them only the right-hand side of the linear system of equations need be recalculated on each iteration. The extra work of creating and decomposing a new matrix on each iteration slows down Newton's method and the Kačanov method. We see, therefore, that the fact that Newton's method has quadratic convergence, while the method of successive approximations and the method of pseudolinear equations have only linear convergence, does not imply that it converges faster than these two methods. That the Kačanov method is the slowest of all the methods is due in part to the fact that for it the matrix must be created and decomposed on each iteration, while the method has an indeterminate theoretical rate of convergence.

The data in Table 5, which treats the same equation as Table 4, show the influence of the parameters of the method of successive approximations and the method of pseudolinear equations. The optimal q_2 is 1.5 and the optimal q, $\hat{q}$ are 0.9, which result in 64% and 13% reductions in CPU time respectively, vs. $q_2 = 1$ and $q = \hat{q} = 1$. While the computing times for widely

Table 2
Numerical results for $f = 0$, $u^{(0)}(x, y) = c\cos(n\pi x)\cos(n\pi y)$.

c_1	c_2	c	Newton	Successive approximations ($q_1 = q_2 = 1$)	Pseudolinear equations ($q = \hat{q} = 1$)
0.01	0.01	0.1	$u^{(3)}$ 25.10 sec converges	$u^{(3)}$ 4.08 sec converges	$u^{(2)}$ 20.66 sec converges
		1	$u^{(24)}$ 201.22 sec converges	$u^{(5)}$ 5.50 sec diverges	$v^{(3)}$ 24.74 sec converges
0.1	0.1	0.1	$u^{(4)}$ 30.94 sec converges	$u^{(4)}$ 4.50 sec converges	$v^{(2)}$ 21.42 sec converges
		1	$u^{(20)}$ 155.60 sec converges	$u^{(3)}$ 3.70 sec diverges	$u^{(4)}$ 25.26 sec converges
1	1	0.1	$u^{(22)}$ 180.34 sec converges	$u^{(5)}$ 5.00 sec diverges	$v^{(3)}$ 24.18 sec converges
		1	$u^{(20)}$ 164.26 sec converges	$u^{(2)}$ 3.06 sec diverges	$v^{(2)}$ 20.84 sec diverges
10	10	0.1	$u^{(20)}$ 162.12 sec converges	$u^{(3)}$ 3.76 sec diverges	$u^{(4)}$ 24.34 sec converges
		1	$u^{(29)}$ 233.56 sec it. limit	$u^{(2)}$ 3.12 sec diverges	$v^{(1)}$ 18.70 sec diverges

differing parameters differ much, both methods are stable with respect to small changes in the parameters.

In Table 6 are shown the storage requirements in words of 50 bits, including system subroutines CHODET and CHOSOL or BADET1 and BASOL1, and the compilation times of the four computer programs. The reason that the program for the method of pseudolinear equations takes more storage and compilation time than do the programs for the other three methods is that in it both Dirichlet and Neumann problems must be treated. Since the program was considered to be experimental, the logic was written in the most straightforward way, that is, separate for the Dirichlet problems and the Neumann problems. Storage requirements and compilation time for this program could have been reduced by combining a good part of the logic for these two types of problems.

The results presented in this section indicate that the method of pseudolinear equations strikes a compromise between how strong the nonlinearities for which the method converges can be and

Table 3
Numerical results for $f(x, y) = c\cos(n\pi x)\cos(n\pi y)$, $u^{(0)} = 0$

c_1	c_2	c	Newton	Kačanov	Successive approximations ($q_1 = q_2 = 1$)	Pseudolinear equations ($q = \hat{q} = 1$)
1	10	10	$u^{(4)}$ 31.90 sec converges	$u^{(5)}$ 38.92 sec converges	$u^{(5)}$ 5.30 sec converges	$u^{(5)}$ 26.02 sec converges
		100	$u^{(29)}$ 229.38 sec it. limit	$u^{(29)}$ 234.00 sec it. limit	$u^{(4)}$ 4.32 sec diverges	$v^{(4)}$ 25.30 sec diverges
10	10	10	$u^{(4)}$ 30.58 sec converges	$u^{(5)}$ 39.92 sec converges	$u^{(6)}$ 6.02 sec converges	$u^{(13)}$ 43.30 sec converges
		20	$u^{(5)}$ 38.32 sec converges	$u^{(21)}$ 164.86 sec converges	$u^{(31)}$ 22.02 sec converges	$u^{(49)}$ 134.00 sec it. limit
		30	$u^{(6)}$ 46.46 sec converges	$u^{(29)}$ 225.80 sec it. limit	$u^{(26)}$ 21.04 sec diverges	$u^{(49)}$ 138.08 sec it. limit
		50	$u^{(29)}$ 233.86 sec it. limit	$u^{(29)}$ 231.16 sec it. limit	$u^{(6)}$ 6.54 sec diverges	$u^{(49)}$ 133.00 sec it. limit
		100	$u^{(29)}$ 235.16 sec it. limit	$u^{(29)}$ 225.50 sec it. limit	$u^{(4)}$ 4.32 sec diverges	$u^{(49)}$ 136.12 sec it. limit
		10^3	$u^{(29)}$ 236.76 sec it. limit	$u^{(29)}$ 223.36 sec it. limit	$u^{(3)}$ 3.80 sec diverges	$v^{(2)}$ 22.22 sec diverges

Table 4
Numerical results for $f(x, y) = cy$, $u^{(0)} = 0$

10	10	1	$u^{(5)}$ 41.76 sec converges	$u^{(11)}$ 87.80 sec converges	$u^{(26)}$ 20.82 sec converges	$u^{(5)}$ 26.66 sec converges
		10	$u^{(8)}$ 66.94 sec converges	$u^{(29)}$ 229.82 sec it. limit	$u^{(3)}$ 4.32 sec diverges	$u^{(49)}$ 137.38 sec it. limit

the speed of convergence. It is expected that the performance of the method of pseudolinear equations (as well as that of the method of successive approximations) would be improved by the use of appropriate variable coefficients instead of constant coefficients.

In terms of overall applicability, the method of pseudolinear equations has one practical advantage over Newton's method that has not been emphasized above. The functions p_i are

Table 5
Numerical results for the method of successive approximations and the method of pseudolinear equations with various parameters for $f(x, y) = cy$, $u^{(0)} = 0$, $c_1 = c_2 = 10$, $c = 1$

q_2 or q, $\hat{q}$	Successive approximations ($q_1 = 1$)	Pseudolinear equations	q_2 or q, $\hat{q}$	Successive approximations ($q_1 = 1$)	Pseudolinear equations
0.3	$u^{(4)}$ 4.54 sec diverges	$u^{(17)}$ 55.18 sec converges	1.2	$u^{(12)}$ 10.98 sec converges	$u^{(5)}$ 26.16 sec converges
0.5	$u^{(6)}$ 6.30 sec diverges	$u^{(8)}$ 34.02 sec converges	1.4	$u^{(9)}$ 8.30 sec converges	$u^{(6)}$ 28.74 sec converges
0.6	$u^{(9)}$ 8.64 sec diverges	$u^{(6)}$ 28.72 sec converges	1.5	$u^{(8)}$ 7.44 sec converges	$u^{(7)}$ 32.22 sec converges
0.7	$u^{(199)}$ 139.80 sec it. limit	$u^{(6)}$ 29.48 sec converges	1.6	$u^{(8)}$ 7.82 sec converges	$u^{(7)}$ 31.56 sec converges
0.8	$u^{(199)}$ 142.76 sec it. limit	$u^{(5)}$ 27.06 sec converges	2.0	$u^{(10)}$ 8.48 sec converges	$u^{(9)}$ 38.88 sec converges
0.9	$u^{(66)}$ 49.18 sec converges	$u^{(4)}$ 22.98 sec converges	3.0	$u^{(15)}$ 11.62 sec converges	$u^{(20)}$ 61.98 sec converges
1.0	$u^{(26)}$ 20.82 sec converges	$u^{(5)}$ 26.66 sec converges	4.0	$u^{(20)}$ 16.44 sec converges	$u^{(49)}$ 133.34 sec it. limit

Table 6
Storage requirements and compilation times of the four computer programs

	Newton	Kačanov	Successive approximations	Pseudolinear equations
Storage required (decimal)	8106 words	8125 words	6009 words	18463 words
Compilation time	18.50 sec	19.62 sec	14.16 sec	35.00 sec

required by the method of pseudolinear equations to be only twice Hölder-differentiable (see Theorem 4.2), while, for Newton's method, they must have roughly one additional order of differentiability (see [2, Ch. 18]).

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